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Essential Singularity in Percolation Problems and Asymptotic Behavior of Cluster Size Distribution

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It is rigorously proved that the analog of the free energy for the bond and site percolation problem on $\mathbb{Z}^\nu$ in arbitrary dimension ν ($\nu > 1$) has a singularity at zero external field as soon as percolation appears, whereas it is analytic for small concentrations. For large concentrations at least, it remains, however, infinitely differentiable and Borel-summable. Results on the asymptotic behavior of the cluster size distribution and its moments, and on the average surface-to-size ratio, are also obtained. Analogous results hold for the cluster generating function of any equilibrium state of a lattice model, including, for example, the Ising model, but infinite-range and n-body interactions are also allowed.

KEY WORDS: Percolation; essential singularity; Ising model; cluster size distribution; central limit theorem.

1. INTRODUCTION

Since being introduced by Broadbent and Hammersley[1] in 1957, the concept of percolation has received increased interest among physicists. The main reason is probably that it provides a well-defined, but nevertheless transparent and intuitively satisfying, geometrical model for spatially random phenomena. On the theoretical side, the problem is very interesting since it leads to the study of nonlocal observables, whereas statistical mechanics is usually concerned only with local observables. On the other hand, the existence of a percolation threshold and the resulting singular behavior of various observables bear a striking similarity with ordinary critical phenomena. This

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analogy has stimulated and guided a large body of the research on this problem.

The connection with second-order phase transitions has been particularly elucidated, following the work of Fortuin and Kasteleyn.[2] It resulted from their work that a quantity $f_p(h)$, an analog of the free energy of one phase, could be defined. It depends on the concentration p and on a parameter h, playing the role of the external magnetic field in ordinary phase transition problems. The analytic behavior of this function with respect to h is connected to the behavior of the moments of the cluster size distribution $\langle |C|^n \rangle$ and of the probability size distribution function P_n. Our purpose here is to study such analyticity properties.

Finally, it is also interesting to study the phenomenon of percolation for systems of interacting particles, such as the Ising model,[3,4] or more generally for an equilibrium state of statistical mechanics. In that case, for any state μ we can study the cluster generating function $f_\mu(h)$, where h is an additional parameter. Analytic properties of $f_\mu(h)$ also allow us to describe in that case the behavior of clusters in the equilibrium state. Our results will extend also to this interacting problem, for the equilibrium state obtained with any type of statistical mechanical potential, of finite or infinite range, two-body or n-body. This extension involves some additional techniques, and results will be paralleled in each section for the noninteracting (usual) and interacting percolation problems.

The paper is organized as follows:

Section 2: The problems are defined, notations stated, and preliminary results established.

Section 3: For small concentrations, $f(h)$ is shown to be analytic at $h = 0$, $\langle |C|^n \rangle$ to behave as $K^n n!$, and P_n to decay exponentially with n.

Section 4: As soon as percolation appears, $f(h)$ is proved to be singular at $h = 0$, $\langle |C|^n \rangle$ is larger than $K^n[n\nu/(\nu - 1)]!$, P_n does not decay exponentially, and $\lim_{n\to\infty}(\langle b \rangle_n/n) = (1 - p)/p$.

Section 5: For large concentrations, $f(h)$ is proved, moreover, to be infinitely differentiable and Borel-summable at $h = 0$, the $\langle |C|^n \rangle$ behave as $K^n[n\nu/(\nu - 1)]!$ and the P_n as $\exp(-\alpha n^{(\nu-1)/\nu})$.

Section 6: These rigorous results find an intuitive basis in a central limit theorem.

We would like to emphasize the fact that results of Section 4 represent a rigorous proof for the free energy of the percolation problem or the cluster generating function of an equilibrium state of the analog of Andreev and Fisher's[5] conjecture of the existence of an essential singularity in the free energy of the Ising model at $h = 0$ below the critical temperature. In this connection, we recall[2] that $f_p(h)$ for the percolation problem is the limit of the free energy of the random cluster model as the parameter tends to one, a model which is identical to the Ising model when the parameter is equal to two.

The proof of our result is by itself of interest, as will be seen in Section 4, because it makes apparent that the clusters in the percolation region have an effective volume, and this phenomenon makes the singularity to appear.

Some of our results have been previously reported in a letter.[6]

2. DEFINITIONS AND PRELIMINARIES

In percolation theory, one considers usually two kinds of processes, the so-called bond percolation and site percolation processes. Although bond percolation is a special case of site percolation, the transformation of one problem into the other involves a change of lattice. For convenience, therefore, we will keep the distinction here.

These processes are usually defined on an infinite lattice (or graph) G, composed of a countable set of vertices $V(G)$ and edges $E(G)$. Here, for convenience, these lattices will be supposed to be regular, i.e., each vertex will have the same finite valence (also called coordination number) and assumed to be translation invariant. These restrictions are not, however, strictly necessary.

In the site problem, a configuration is defined as a subset $V' \subset V(G)$ of the vertices (or sites) that are considered as occupied, the remaining ones $V(G)\backslash V'$ being considered as vacant. A probability distribution on this configuration space is given and determined as usual by the set of probability of local events $\{\mu_\Lambda(X)\}$, where $\mu_\Lambda(X)$ is the probability that in the finite subgraph Λ all the vertices of the set X are occupied, whereas those of the complement $V(\Lambda)\backslash X$ are vacant. In the usual percolation problem, the vertices are chosen to be occupied or vacant independently of each other with probability p and $q = 1 - p$. Hence

$$\mu_\Lambda(X) = p^{|X|}q^{|V(\Lambda)\backslash X|} \tag{1}$$

where X denotes the number of points of the set X.

It is quite possible, however, and it is necessary for the study of the clusters of, say, the Ising model, to consider the more general case where the occupied sites interact and the probability distribution $\mu_\Lambda(X)$ is an equilibrium state of a lattice model at a given temperature and chemical potential. This more general situation has been discussed recently in the case of the Ising model.[3,4] It is often believed that such a study can only be achieved if the graph defining the connectivity and the graph of the nonzero interactions are the same. This idea follows from the a priori that different clusters should be noninteracting. As a matter of fact, we mention that such a restriction is not necessary and in our work the potential is completely independent of the graph that determines the connectivity. This will allow us to treat measures of probability that are equilibrium states of systems with infinite-range or many-body interactions, and not only those of finite range. In all these cases, the study of percolative processes for such distribution measures leads to interesting results concerning the typical configurations, behavior of the cluster distribution, nucleation, and so on.

As in statistical mechanics, one introduces local observables, which are defined as functions on the configuration space, such that $f(V') = f(V' \cap V(\Lambda))$ for some finite subgraph Λ and $\sup_{V'}|f(V')| < \infty$. The expectation value of such an observable is given by

$$\langle f \rangle = \sum_{X \subset \Lambda} f(X)\mu_\Lambda(X) \tag{2}$$

In percolation, however, contrary to usual statistical mechanics, one is not interested in local observables (whose expectation values are trivial in the noninteracting case), but in a special class of nonlocal observables, the cluster observables. They are introduced in the following way: In a given configuration, the subset $V' \subset V(G)$ consisting of all the occupied vertices defines a section graph $G' = (V', E')$, where E' consists of all edges of E with both vertices in V'. Each connected component of G' defines a maximal connected set of occupied sites. A simple cluster observable is the characteristic function $\gamma_X{}^f$ (or $\gamma_X{}^\infty$) of the event: "The finite set of vertices X belongs to the same finite (or infinite) cluster." (We recall that the characteristic function of some event is the function over the configuration that takes the value one if the configuration satisfies the event and the value zero otherwise.) General cluster observables are obtained by taking linear combinations and products of the simple cluster observables. These observables can be obtained as limits of local observables for regular lattices at least.[2] This allows us, in principle, to compute their expectation value.

The bond percolation problem is defined in the same way, but this time a configuration is given by the subset $E' \subset E(G)$ of the edges of the lattice that is considered to be occupied, the remaining one $E(G)\backslash E'$ being vacant. In the independent case, the probability distribution is given by $\mu_\Lambda(E') = p^{|E'|}q^{|E(\Lambda)\backslash E'|}$, where p and $q = 1 - p$ are the probabilities that an edge is occupied or vacant. Then clusters and cluster observables can be defined in a similar way as in the site problem.

In this paper, we will mainly study the analog of thermodynamic quantities for the percolation problem, namely the cluster generating function, and its various derivatives. If P_n is the cluster size distribution function, which means that P_n is the probability that a given point belongs to a cluster of exactly n vertices, then the cluster generating function is defined by

$$f(h) = \sum_{n=1}^{\infty} \frac{1}{n} P_n e^{-hn} \tag{3}$$

This series is always convergent when $h \geqslant 0$, because $\sum_{n=1}^{\infty} P_n \leqslant 1$. The parameter h is called the external field.

The analogy between this quantity and the free energy of one phase in ordinary lattice models of ferromagnetism was first put on a firm basis by Fortuin and Kasteleyn[2] (see also Ref. 7). They proved that $f(h)$, for the usual bond model, could be obtained as the limit of the free energy of the ferromagnetic Potts model with k components, when k tends to 1, the magnetic field staying positive. The parameter h of the percolation problem is then identified as the magnetic field of the Potts model, whereas $p = 1 - e^{-\beta J}$, where J is the coupling constant. This correspondence can be extended to the site percolation model.[8] In the interacting case, however, h should be clearly distinguished from the magnetic field, and $f(h)$, which we will denote in this case by $f_\mu(h)$ (by reference to the equilibrium state μ) can only be considered as the cluster generating function.

Knowing the cluster generating function, one can compute various quantities, namely the cluster size distribution function P_n, which is of central interest in percolation, and its moments.

First of all, in the variable $z = e^{-h}$, we have

$$\frac{d^k}{dz^k} f(z)\bigg|_{z=0} = (k-1)!\, P_k \tag{4}$$

On the other hand, we have, at least formally, up to now, when $k > 2$,

$$(-1)^k \frac{d^k}{dh^k} f(h)\bigg|_{h=0} = \langle n^{k-1} \rangle \tag{5}$$

In particular, $f(0)$ is the mean number of clusters per site, and

$$-f^{(1)}(0) = P_f = \rho_\mu - P_\infty \tag{6}$$

where P_∞ is the percolation probability, i.e., the probability that the origin belongs to an infinite cluster, P_f is the probability that it belongs to a finite one, and ρ_μ is the probability that the origin is occupied; ρ_μ reduces to p in the noninteracting case. The second derivative $f^{(2)}(0)$ is the average size of finite clusters, and the following derivatives are the further moments of the P_n.

We are now in a position to state the following preliminary results:

1. The function $f(h)$ is analytic for $\operatorname{Re} h > 0$ and the series $f(z)$ is absolutely convergent for $|z| \leqslant 1$.

This follows immediately from the fact that $\sum_{n \geqslant 1} P_n \leqslant 1$.

2. The analyticity of $f(h)$ at $h = 0$ is equivalent to the convergence (and analyticity) of $f(z)$ in a circle of radius larger than 1, to the exponential decay of the P_n with n, and to a bound of the form $K^k k!$ for the moments $\langle |C|^k \rangle$.

In particular, if the series (3) is divergent for any $h < 0$, then $f(h)$ is singular at $h = 0$.

This follows from the positivity of the coefficients of the series (3): It implies that the first singularity of $f(z)$ appears for z real, positive. Analyticity in h at $h = 0$ implies, then, analyticity of $f(z)$ for $|z| < z_0$, $z_0 > 1$. It implies in turn that P_n decays exponentially with n, and that the moments are finite and bounded by $K^k k!$. The converse also holds, as can be easily seen.

In conclusion, the interesting question is that of the possible analyticity or singularity at $h = 0$, which is related to the behavior of $\langle |C|^n \rangle$ and to the exponential decay of the P_n. By analogy with magnetic systems, we might expect that in the percolation problems, $f(h)$ is analytic at $h = 0$ outside the percolative region, whereas inside the percolative region various conjectures are possible, one based on mean-field-type arguments, from which $f(h)$ should stay analytic at $h = 0$, the other based on an analog of Andreev and Fisher's conjecture[5] for the liquid–gas transition, arguing for a singularity. The latter picture will be proved to be correct.

3. ANALYTICITY FOR LOW CONCENTRATION

Let us first consider the usual percolation problem. We will show that for $p < p_0 < p_c$, p_c denoting the percolation threshold over which percolation appears, the function $f_p(h)$ is analytic at $h = 0$. This result was already known by Lieb[9] from arguments on analytic functions of several variables. We give here a direct proof.

In the independent case, $f(h)$, denoted $f_p(h)$, can be written explicitly as

$$f(h) = \sum_{C \ni \{0\}} \frac{e^{-h|C|}}{|C|} p^{|C|} q^{|\partial C|} = \sum_{C_1} e^{-h|C|} p^{|C|} q^{|\partial C|} \tag{7}$$

where the sum runs over all the finite clusters C containing the origin in the first expression, and over all possible shapes of finite clusters in the second one. $|C|$ denotes the size of the cluster and $|\partial C|$ the size of its boundary, i.e., the number of points of $V(G)\backslash C$ that are neighbors to some point of C in the site case, and the number of edges of $E(G)\backslash C$ that have at least one vertex in common with C in the bond case.

It is known (see, for example, Ref. 10) that the number of clusters of size n is less than K^n, where K is some constant depending on the lattice. Henceforth

$$P_n \leqslant p^n K^n \tag{8}$$

and $f_p(h)$ is analytic in the region $\operatorname{Re} h \geqslant -\log pK$ and hence at $h = 0$ for $p < K^{-1}$.

On the other hand, it is easy to see, by restricting the sum (7), for example, to the chains, that for any p

$$P_n \geqslant p^n q^{(z-2)n+2} \tag{9}$$

where z is the coordination number of the lattice, and so P_n behaves exponentially with n at small concentration.

We want now to indicate a simple method to obtain a better range of concentrations for which $f_p(h)$ is analytic at $h = 0$. A related one has been worked out by Schwartz.[11]

Let us denote by $a(n, b)$ the number of clusters of size n and boundary size b. Then

$$P_n(p) = p^n \sum_b a(n, b) q^b \tag{10}$$

and

$$\frac{d \log(P_n/p^n)}{d \log q} = \frac{\sum_b b a(n, b) q^b}{\sum_b a(n, b) q^b} \tag{11}$$

If we consider the site problem, we have, moreover,

$$b \leqslant n(z - 2) + 2 \tag{12}$$

Then

$$\frac{d \log(P_n/p^n)}{d \log q} \leqslant n(z - 2) + 2 \tag{13}$$

Integrating this inequality and using that $\log P_n \leqslant 0$, we obtain for $p < p'$

$$\log P_n(p) \leqslant [(z - 2)n + 2] \log(q/q') + n \log(p/p') \tag{14}$$

Choosing $p' = 1/(z - 1)$, $q' = 1 - p' = (z - 2)/(z - 1)$, which maximizes $p'(q')^{z-2}$, we get

$$P_n(p) \leqslant \left(\frac{z-1}{z-2} q\right)^2 \left[\frac{(z-1)^{z-1}}{(z-2)^{z-2}} pq^{z-2}\right]^n \tag{15}$$

from which analyticity at $h = 0$ follows for $p < 1/(z - 1)$. Note that $1/(z - 1)$ is the critical probability for percolation on the Bethe lattice.

On the other hand, (9) ensures that

$$P_n(p) \geqslant q^2[pq^{z-2}]^n \sum_b a(n, b) \geqslant q^2[pq^{z-2}]^n \tag{16}$$

Finally, in the bond problem, the same method applies, the basic inequality (12) becoming

$$b \leqslant n(z - 2) + z \tag{17}$$

We can then state:

Theorem 1. For the site and bond percolation problems with coordination number z, the free energy satisfies the following properties:

(i) $f_p(h)$ is analytic at $h = 0$ for $p < 1/(z-1)$.

(ii) P_n decreases exponentially with n for $p < 1/(z-1)$ and, more precisely, if $\alpha = 2$ or z for the site or bond problem respectively,

$$q^\alpha[pq^{z-2}]^n \leqslant P_n \leqslant \left(\frac{z-1}{z-2}q\right)^\alpha \left[\frac{(z-1)^{z-1}}{(z-2)^{z-2}} pq^{z-2}\right]^n \tag{18}$$

(iii) $\langle |C|^n \rangle$ satisfies the following inequalities:

$$A_1 K_1{}^n n! \leqslant \langle |C|^n \rangle \leqslant A_2 K_2{}^n n!$$

with A_1, K_1, A_2, and K_2 obtained from (18).

Let us now turn our attention to the percolation problem in the interacting case. For simplicity we will not bother here with the size of the domain of analyticity. We will use the lattice gas interpretation. The probability $P(C)$ of having some cluster C is, by definition, the probability $P(C$ occupied and ∂C empty). But

$$P(C \text{ occupied and } \partial C \text{ empty}) \leqslant P(C \text{ occupied}) = \rho(C) \tag{19}$$

where $\rho(C)$ is the correlation function for the set C. Furthermore, it is known [see, for example, Ref. (12)] that for small concentration ρ, one has

$$\rho(C) < A(\rho)^{|C|} \tag{20}$$

and $A(\rho)$ tends to zero as ρ tends to zero.

Hence, the number of clusters of size n being bounded by K^n, P_n is less than $[KA(\rho)]^n$ and $f_\mu(h)$ is analytic at $h = 0$ for ρ small enough.

This result depends only on the bound (20) and is then independent of the range of the interaction or the presence of many-body potentials in the interaction, as long as

$$\sum_{\substack{X \subset V(G) \\ X \ni \{0\}}} |\Phi(X)| < \infty$$

We can then state:

Theorem 2. For the interacting percolation problem, the cluster generating function $f_\mu(h)$ is analytic in h at $h = 0$ and $P_n(\mu)$ decays exponentially when the concentration ρ is small enough: $\rho < \rho_0$.

4. SINGULARITY IN THE PERCOLATIVE REGION

The situation when there is percolation appears to be much more interesting. As stated, we will prove that there is a singularity of $f_p(h)$ or $f_\mu(h)$ whenever there is percolation. The proof will use a new description of $f(h)$ and its moments as well as P_n. It will allow us to understand their properties. In fact, the crucial and most interesting point will be the proof that the clusters have an effective volume in the percolative region.

We will associate to each cluster C a contour γ, in the following way: First consider, for simplicity, the cubic lattices $\mathbb{Z}^\nu$ (see Fig. 1). We imbedded our lattice in $\mathbb{R}^\nu$ in the trivial manner, and given any cluster, we draw around each vertex x of the cluster the 2ν faces of the unit cube centered at x. We suppress, then, the faces that occur twice. The closed polyhedron obtained in this way is called $\Gamma(C)$. Each face of $\Gamma(C)$ separates a point x of C and a point y of $\mathbb{Z}^\nu \backslash C$. Along a $(\nu - 2)$-dimensional edge of $\Gamma(C)$, either two or four faces meet. In the latter case we slightly deform the polyhedron, "chopping off" the edge from the cubes containing a point of C. When this is done, $\Gamma(C)$ splits into connected components $\gamma_1, \ldots, \gamma_r$, which we call contours. Among all these contours there is one and only one that is outer; we will call it γ, and so to each cluster we have associated a contour γ.

For more general lattices, like those used in solid-state physics (triangular, fcc), instead of a unit cube, we can take the Wigner–Seitz cell and then carry out the same construction.

Let us now first investigate the noninteracting site problem. We introduce some useful notations (see Fig. 2): $V(\gamma)$ will denote the set of points of the lattice inside γ, $\Delta\gamma$ the set of points of $V(\gamma)$ neighbors to some point of the outside of γ, and $\Theta(\gamma)$ is $V(\gamma)\backslash\Delta\gamma$; $\partial\gamma$ will be the set of points outside γ neighbors to some point of $V(\gamma)$.

Then a cluster decomposes naturally into two parts: $C = \Delta\gamma \cup E$, where E is a set of vertices in $\Theta(\gamma)$ that are all connected to the boundary of $\Theta(\gamma)$ by a path of occupied sites in the site problem and of occupied bonds in the bond

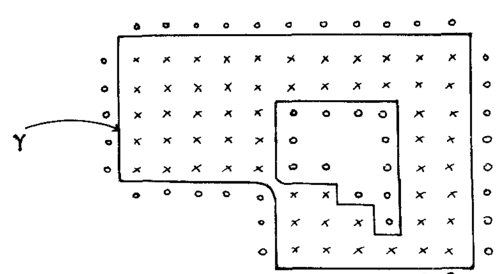

Fig. 1. Association of a contour γ to a cluster C on $\mathbb{Z}^2$. The crosses represent the vertices of C.

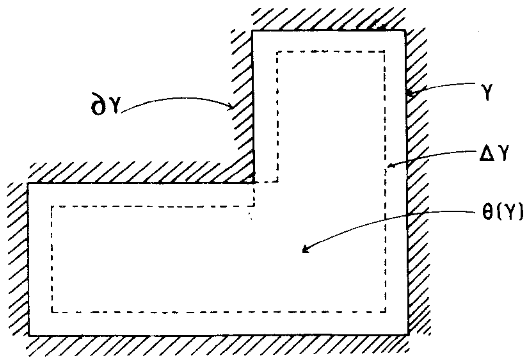

Fig. 2

problem, and E is such that the set of points of E and $\Delta\gamma$ is connected. Note that E can be possibly empty, if $\Delta\gamma$ is a connected set.

We are now going to "separate" the contribution of each cluster into a surface term and a volume term corresponding to the interior of the cluster.

By virtue of the positivity of the terms of the series in (7), we can rearrange the terms. Then

$$f_p(h) = \sum_{\gamma_1} p^{|\Delta\gamma|} q^{|\partial\gamma|} e^{-h|\Delta\gamma|} \sum_{\substack{E \subset \Theta(\gamma) \\ (E,\Delta\gamma)^c}} e^{-h|E|} p^{|E|} q^{|\partial E \cap \Theta(\gamma)|} \tag{21}$$

where the first sum runs over all possible shapes of contours, and the second one over all subsets E included in $\Theta(\gamma)$ such that all points of E and of $\Delta\gamma$ are connected, including possibly the empty set. Here and in all the following, one can replace the summation over γ_1 by a summation over all contours encircling the origin:

$$\sum_{\gamma_1} \equiv \sum_{\gamma, \{0\} \in V(\gamma)} 1/|V(\gamma)| \tag{22}$$

Now the sum over E on the right-hand side of (21) can be reexpressed as $\langle \chi^c(\Delta\gamma) \exp(-h \sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma}) \rangle_{\Theta(\gamma)}$, that is, an average value over the configurations of $\Theta(\gamma)$. Here $\chi^c(\Delta\gamma)$ denotes the characteristic function of the event "the set formed by the points of $\Delta\gamma$ and the occupied points of $\Theta(\gamma)$ that are connected to $\Delta\gamma$ is connected," and then restricts the average value to the configurations satisfying that condition; on the other hand, $\chi_x^{\Delta\gamma}$ is the characteristic function of the event "x is connected to $\Delta\gamma$," so $\sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma}$ gives for each configuration the number of points connected to the boundary. Hence we get

$$f_p(h) = \sum_{\gamma_1} p^{|\Delta\gamma|} q^{|\partial\gamma|} [\exp(-h|\Delta\gamma|)] \left\langle \chi^c(\Delta\gamma) \exp\left(-h \sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma}\right) \right\rangle_{\Theta(\gamma)} \tag{23}$$

As a matter of fact, when $\Delta\gamma$ is by itself a connected set, $\chi^c(\Delta\gamma) \equiv 1$, and then, if $\sum_{\gamma'}$ denotes the summation over all contours such that $\Delta\gamma$ is a connected set, we get

$$f_p(h) \geqslant \sum_{\gamma_1'} p^{|\Delta\gamma|}q^{|\partial\gamma|}[\exp(-h|\Delta\gamma|)]\left\langle \exp\left(-h \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right)\right\rangle_{\Theta(\gamma)} \tag{24}$$

for h real and then we get by Jensen's inequality

$$f_p(h) \geqslant \sum_{\gamma_1'} p^{|\Delta\gamma|}q^{|\partial\gamma|}[\exp(-h|\Delta\gamma|)] \exp\left(-h\left\langle \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right\rangle_{\Theta(\gamma)}\right) \tag{25}$$

Moreover, it is clear that $\langle\chi_x^{\Theta(\gamma)}\rangle_{\Theta(\gamma)} \geqslant P_\infty$, where P_∞ denotes the percolation probability, and then if we restrict ourselves to h real, negative, $h = -\epsilon$, we have

$$f_p(h) \geqslant \sum_{\gamma_1'} p^{|\Delta\gamma|}q^{|\partial\gamma|} \exp[\epsilon(|\Delta\gamma| + |\Theta(\gamma)|P_\infty)] \tag{26}$$

Suppose now that there exists a sequence of contours γ_n such that $\Delta\gamma$ is a connected set, and, moreover,

$$\lim_{n\to\infty} |\partial\gamma_n| = \infty, \qquad \lim_{n\to\infty} \frac{|\partial\gamma_n|}{|\Delta\gamma_n|} = a, \qquad \lim_{n\to\infty} \frac{|\partial\gamma_n|}{|\Theta(\gamma_n)|} = 0 \tag{27}$$

then for any $\epsilon, f_p(-\epsilon)$ will be infinite. This in turn would imply the singularity of $f_p(h)$ at $h = 0$, as indicated in the second preliminary remark of Section 2.

For the simple cubic lattice $\mathbb{Z}^\nu$, it is not difficult to find such a sequence when $\nu > 1$. They are just the cubes. For more complicated lattices, they can be found by inspection. For lattices that contain $\mathbb{Z}^\nu$ (for example, nearest neighbors plus next nearest neighbors) we can again take cubes for γ_n, as long as the coordination number of each site is finite. In some cases, in order to apply our arguments, it is useful to separate not only the contribution of the set $\Delta\gamma$, but also that of some other set of points. For example, in the hexagonal lattice, it is useful to consider, besides $\Delta\gamma$, the set $\Delta'\gamma$ of points inside $V(\gamma)$ that are second nearest neighbors to the outside of γ. Then by restricting the configurations to those such that $\Delta'\gamma$ is completely occupied and choosing as a sequence of contours the regular hexagons, the proof goes quite similarly as previously. Alternatively, this can be worked out directly through the use of the Wigner–Seitz cell construction for γ and then defining $|\Delta\gamma|$ as the set of points of $V(\gamma)$ contained in some cell intersecting γ.

Note, however, that such a sequence does not exist in the case of a Bethe tree or expanded cactusses, which explains why $f_p(h)$ is analytic at $h = 0$ for $p > p_c$ in these cases,[13] whereas we can prove here the nonanalyticity for realistic lattices.

Furthermore, we can say more about the singularity by looking at the derivatives of $f(h)$ at $h = 0$, i.e., up to a sign, the moments of the cluster size distribution $\langle |C|^l \rangle$.

Applying the same ideas to them, and by virtue of the positivity of their representative series, we get

$$\langle |C|^l \rangle = \sum_{\gamma_1} p^{|\Delta\gamma|} q^{|\partial\gamma|} \sum_{\substack{E \subset \Theta(\gamma) \\ (E, \Delta\gamma)^c}} (\Delta\gamma + |E|)^l p^{|E|} q^{|\partial E \cap \Theta(\gamma)|} \tag{28}$$

But Eq. (28) can be rewritten as

$$\langle |C|^l \rangle = \sum_{\gamma_1} p^{|\Delta\gamma|} q^{|\partial\gamma|} \left\langle \chi^c(\Delta\gamma) \left[|\Delta\gamma| + \sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma} \right]^l \right\rangle_{\Theta(\gamma)} \tag{29}$$

However, the functions involved in the average value are certainly increasing over the configurations of $\Theta(\gamma)$. [We recall that a function f is said to be increasing over configurations if $f(X) \geqslant f(X')$ whenever the set of occupied points in the configuration X contains the set of occupied points of the configuration X'.] In percolation, Harris' inequality[14] (see also Ref. 2) tells us that if $f_1, ..., f_n$ are n increasing functions over configurations, then

$$\left\langle \prod_{i=1}^{n} f_i \right\rangle \geqslant \prod_{i=1}^{n} \langle f_i \rangle$$

In our situation this implies that

$$\begin{aligned} &\left\langle \chi^c(\Delta\gamma) \left[|\Delta\gamma| + \sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma} \right]^l \right\rangle_{\Theta(\gamma)} \\ &\qquad \geqslant \langle \chi^c(\Delta\gamma) \rangle_{\Theta(\gamma)} \left[\left\langle \left(\Delta\gamma + \sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma} \right) \right\rangle_{\Theta(\gamma)} \right]^l \end{aligned} \tag{30}$$

and since $\langle \chi_x^{\Delta\gamma} \rangle_{\Theta(\gamma)} \geqslant P_\infty$,

$$\langle |C|^l \rangle \geqslant \sum_{\gamma_1} p^{|\Delta\gamma|} q^{|\partial\gamma|} \langle \chi^c(\Delta\gamma) \rangle [|\Delta\gamma| + P_\infty |\Theta(\gamma)|]^l \tag{31}$$

We can now look again at the contribution of the contour γ_n introduced before, which leads to the lower bound

$$\langle |C|^l \rangle \geqslant K_1{}^l \left(\frac{\nu}{\nu - 1} l \right)!$$

As a matter of fact, this proof shows that over the percolation threshold, the clusters have an effective volume. The divergence follows from this fact and from the existence in realistic lattices of contours whose volume grows faster than the surface, in contrast to the Bethe-type lattices, for which clusters also have an effective volume, but in which, by geometric restrictions, this volume is of the order of the surface.

Let us now briefly come to the bond problem. The same methods will apply with minor differences. The only change follows from the fact that given a connected set of sites, there are possibly various clusters of bonds compatible with that cluster of sites. Hence, after choosing the set of sites of the cluster, one has to sum over bond configurations compatible with it. Precisely, one can express $f_p(h)$ as

$$f_p(h) = \sum_{\gamma_1} q^{|\gamma|} \sum_{B \subset \mathscr{B}(\Delta\gamma)} p^{|B|} e^{-h|B|} \sum_{\substack{B' \subset \mathscr{B}(\Theta(\gamma)) \\ (\Delta\gamma, B')^c}} e^{-h|B'|} p^{|B'|} q^{|\partial B' \cap \mathscr{B}(\Theta(\gamma))|} \tag{32}$$

$$= \sum_{\gamma_1, B \subset \mathscr{B}(\Delta\gamma)} q^{|\gamma|} p^{|B|} e^{-h|B|} \left\langle \chi^c(\Delta\gamma) \exp\left(-h \sum_{b \in \mathscr{B}(\Theta(\gamma))} \chi_b^{\Delta\gamma}\right)\right\rangle_{\Theta(\gamma)} \tag{33}$$

where $|\gamma|$ denotes the surface of the contour γ, that is, the number of bonds joining one point of $\Delta\gamma$ with one point of $\partial\gamma$; the sum over B runs over all bond configurations among the set $\mathscr{B}(\Delta\gamma)$ of bonds between points of $\Delta\gamma$; the sum over B' runs over all bond configurations among the set $\mathscr{B}(\Theta(\gamma))$ of bonds between points of $\Theta(\gamma)$ such that the points of $\Delta\gamma$ and the bonds B' are connected; and $\chi_b^{\Delta\gamma}$ denotes the characteristic function of the event "the bond b is connected to the boundary of $\Theta(\gamma)$."

The moments can be expressed also in the following formula:

$$\langle |C|^l \rangle = \sum_{\gamma_1, B \subset \mathscr{B}(\Delta\gamma)} q^{|\gamma|} p^{|B|} \left\langle \chi^c(\Delta\gamma) \left(|B| + \sum_{b \in \Theta(\gamma)} \chi_b^{\Delta\gamma} \right)^l \right\rangle_{\Theta(\gamma)} \tag{34}$$

Then the proofs go as in the site problem.

We can now state the results:

Theorem 3. For the (noninteracting) site and bond problems on $\mathbb{Z}^\nu$, $\nu > 1$, the following results hold for $p > p_c$:

(i) The free energy $f_p(h)$ is singular at $h = 0$ and the P_n do not decay exponentially.

(ii) The moments of the cluster size distribution satisfy for $p > p_c$

$$\langle |C|^l \rangle \geqslant K_1{}^l \left(\frac{\nu}{\nu - 1}\, l \right)!$$

Analogous results hold also for the other realistic lattices as discussed in the text.

A consequence of these results will be made explicit later in a corollary; however, we first make two remarks:

Remark 1. If the P_n behave as $\exp(-\alpha n^\xi)$, our result on the behavior of $\langle |C|^l \rangle$ implies that for $p > p_c$ one has $P_n > \exp(-\alpha n^{(\nu-1)/\nu})$. This result was proposed by Stauffer[15] in the two-dimensional case and Flamang[16] in the three-dimensional case, on the basis of an analysis of numerical studies. We will see in Section 5 that for large concentrations we can prove that the P_n in fact decay exactly like $\exp(-\alpha n^{(\nu-1)/\nu})$, that is, we can prove upper and lower bounds of this type for the P_n.

Remark 2. We have proved that $f_p(h)$ is singular at $h = 0$ whenever there is percolation, that is, for $p > p_c$. It is not hard to see that $f_p(h)$ is also singular at $h = 0$ when $p = p_c$. In fact, suppose $f_{p_c}(h)$ to be analytic at $h = 0$, which implies (from the preliminaries in Section 2) that for some positive ϵ one should have

$$\sum_{n \geqslant 1} e^{\epsilon n} P_n(p_c) < \infty$$

But on the other hand, for any complex p' we certainly have

$$\begin{aligned}
|P_n(p')| &\equiv \left| \sum_{\substack{C \ni \{0\} \\ |C| = n}} p'^{|C|}(1 - p')^{|\partial C|} \right| \\
&= \left| \sum_{\substack{C \ni \{0\} \\ |C| = n}} p_c^{|C|}(1 - p_c)^{|\partial C|} \left(1 + \frac{p' - p_c}{p_c}\right)^{|C|} \left(1 - \frac{p' - p_c}{1 - p_c}\right)^{|\partial C|} \right| \\
&\leqslant \sum_{\substack{C \in \{0\} \\ |C| = n}} p_c^{|C|}(1 - p_c)^{|\partial C|} \left(1 + \frac{|p' - p_c|}{p_c}\right)^n \left(1 + \frac{|p' - p_c|}{1 - p_c}\right)^{n(z-2)+2} \\
&\leqslant \left(1 + \frac{|p' - p_c|}{p_c}\right)^n \left(1 + \frac{|p' - p_c|}{1 - p_c}\right)^{n(z-2)+2} P_n(p_c)
\end{aligned} \tag{35}$$

This last expression is less than $e^{\epsilon n/2} P_n(p_c)$ if $|p' - p_c| < \alpha$, for some α sufficiently small. Hence for $|p' - p_c| < \alpha$ the series $\sum_n P_n(p')$ would be absolutely convergent and would define an analytic function; this is impossible since for p' real, $\sum_n P_n(p')$ is equal to p' for p' less than p_c and equal to $p' - P_\infty(p')$ for p' larger than p_c.

An interesting consequence of Theorem 3 concerns the limiting value of the surface-to-size ratio for clusters of size n, namely $\langle b \rangle_n / n$, where $\langle b \rangle_n$ is the average boundary size for clusters of size n. Such a result was proposed at p_c by Stauffer[17] from a scaling hypothesis and obtained at p_c by Reich and Leath[17] in an approximation of P_n through an integral and the use of steepest descent method. On the other hand, a derivation was given for $p > p_c$ by Stauffer[18] using our Theorem 3, but with the implicit assumptions that $\lim_n (1/n) \log P_n$ exists and that limits over n and derivations can be commuted. We give here rigorous results on the limit of $\langle b \rangle_n / n$:

Corollary. For $p > p_c$ one has

$$\lim_{n\to\infty} \frac{\langle b\rangle_n}{n} = \frac{1-p}{p}$$

where $\langle b\rangle_n$ is the average boundary size for clusters of n sites. At p_c one has

$$\liminf_n \frac{\langle b\rangle_n}{n} \geqslant \frac{1-p_c}{p_c}$$

In order to prove this corollary, let us first prove the following lemma:

Lemma. For any p, one has:

(i) $P_{n+m}/(n+m) \geqslant (P_n/n)(P_m/m)$.
(ii) $g_n = (1/n)\log P_n/n$ has a limit when n goes to infinity.

For ease of description, we restrict ourselves here to $\mathbb{Z}^2$, but the proof applies to any lattice. Let us consider two clusters $C^{(1)}$ and $C^{(2)}$ of respective size n_1 and n_2; among the points of $C^{(1)}$ that are farther to the right, we call $x(C^{(1)})$ the highest one, and analogously, among the points of $C^{(2)}$ that are farther to the left, we denote by $y(C^{(2)})$ the lowest one.

We now translate $C^{(2)}$ on the lattice in order that $y(C^{(2)})$ becomes the nearest neighbor to $x(C^{(1)})$ on the right of $x(C^{(1)})$. By virtue of the construction, the points of $C^{(1)}$ and $C^{(2)}$ do not overlap and we can identify the figure obtained with a cluster C of size $n_1 + n_2$. Moreover, the probability $P(C)$ of this cluster satisfies

$$P(C) = P(C^{(1)})P(C^{(2)})/q^a \tag{36}$$

where a is the number of points of the boundary of $C^{(1)}$ and $C^{(2)}$ that overlap after the translation. Hence

$$P(C) \geqslant P(C^{(1)})P(C^{(2)}) \tag{37}$$

Now from another pair of clusters $C'^{(1)}$ and $C'^{(2)}$ of respective size n_1 and n_2, we obtain by our construction a cluster C' with a different shape than C as soon as $C'^{(1)}$ is different from $C^{(1)}$ or $C'^{(2)}$ different from $C^{(2)}$. We see that in this way we have realized an injection from $\{C\}_{n_1} \times \{C\}_{n_2}$ into $\{C\}_{n_1+n_2}$, where $\{C\}_n$ denotes the set of the different shapes of clusters of size n. Hence, using (37), we get

$$\begin{aligned}\frac{1}{n_1+n_2} P_{n_1+n_2} &\equiv \sum_{\substack{C\ni\{0\}\\ |C|=n_1+n_2}} \frac{P(C)}{n_1+n_2} \geqslant \sum_{\substack{C_1\\ |C|=n_1+n_2}} P(C) \\ &\geqslant \sum_{C_1^{(1)},|C^{(1)}|=n_1} P(C^{(1)}) \sum_{C_1^{(2)},|C^{(2)}|=n_2} P(C^{(2)}) \equiv \frac{P_{n_1}}{n_1}\frac{P_{n_2}}{n_2}\end{aligned} \tag{38}$$

where, as previously, summations indexed by a subscript 1 run over the possible shapes of clusters for a given size. This concludes part (i) of the lemma

Inequality (38) can be rewritten as

$$(n_1 + n_2)g_{n_1+n_2} \geqslant n_1 g_{n_1} + n_2 g_{n_2} \tag{39}$$

So the functions ng_n of n are upper-additive and the g_n are bounded above and below ($P_n < 1$ so $g_n \leqslant 0$ and $P_n \geqslant e^{-\alpha n}$ so $g_n \geqslant -\alpha$). This in turn implies (e.g., Ref. 12, Chapter 7) that the g_n have a limit as n goes to infinity. This concludes part (ii) of the lemma.

We turn now to the proof of the corollary. It follows from Theorem 3 that for $p > p_c$, $\limsup_n g_n = 0$ and hence by the lemma, $\lim_n g_n = 0$. Thence

$$\lim_n \frac{1}{n} \log \frac{P_n}{p^n} = -\log p \tag{40}$$

On the other hand, formula (11) yields

$$\frac{d}{d \log q} \frac{1}{n} \log \frac{P_n}{p^n} = \frac{\langle b \rangle_n}{n} \tag{41}$$

Furthermore, the functions $(1/n) \log(P_n/p^n)$ are convex with respect to $\log q$, as can be seen by computing the second derivative, which can be reexpressed as

$$\frac{d^2}{(d \log q)^2} \frac{1}{n} \log \frac{P_n}{p^n} = \frac{1}{n} \langle (b - \langle b \rangle_n)^2 \rangle_n \tag{42}$$

The functions $(1/n) \log(P_n/p^n)$ are then convex and differentiable in the variable $\log q$, they have a limit when n goes to infinity, and this limit, $-\log p$, is differentiable. Hence by Griffith's lemma their derivatives have a limit and this limit is the derivative of the limit:

$$\lim_{n\to\infty} \frac{\langle b \rangle_n}{n} = \frac{d}{d \log q} (-\log p) = \frac{1-p}{p} \tag{43}$$

The inequality at p_c is obtained in a similar way, using Remark 2 after Theorem 3. However, the equality, if true at p_c, would need some more results. In Ref. 18, scaling assumptions were used.

We want now to discuss the question of the analytic behavior of the cluster generating function in the percolative region in the case of the interacting percolation problem. For simplicity we will restrict our attention to the interacting site problem over $\mathbb{Z}^\nu$, and the results can be extended as previously to the other realistic lattices; in particular, in the case of finite-range interactions, one can choose the graph of nonzero interactions as the graph defining the connectivity.

Our probability measure will be the equilibrium measure of some statistical mechanical system, corresponding to some boundary conditions, to some chemical potential μ, and to some translation-invariant interaction potential Φ, such that

$$\sum_{\substack{X \subset \mathbb{Z}^\nu \\ X \ni \{0\}}} |\Phi(X)| = D < \infty$$

The temperature is included in the definition of the potential.

Now the probability of occurrence of some finite cluster $P(C)$ can be expressed as

$$P(C) = P(\gamma)P(C|\gamma) \tag{44}$$

where $P(\gamma)$ is the probability that $\Delta\gamma$ is occupied and $\partial\gamma$ empty, and $P(\cdot|\gamma)$ is the conditional probability with respect to the event γ.

Hence, following ideas similar to previous ones, we can write

$$f_\mu(h) = \sum_{\gamma_1} e^{-h|\Delta\gamma|}P(\gamma) \sum_{\substack{C \text{ compatible} \\ \text{with } \gamma}} P(C|\gamma)e^{-h(|C|-|\Delta\gamma|)} \tag{45}$$

$$= \sum_{\gamma_1} e^{-h|\Delta\gamma|}P(\gamma) \sum_{\substack{E \subset \Theta(\gamma) \\ (E,\Delta\gamma)^c}} P(E, \overline{\partial E \cap \Theta(\gamma)}|\gamma)e^{-h|E|} \tag{46}$$

where $P(X, \bar{Y})$ denotes the probability that X is occupied and Y empty. Now

$$f_\mu(h) = \sum_{\gamma_1} P(\gamma)[\exp(-h|\Delta\gamma|)]\left\langle \chi^c(\Delta\gamma) \exp\left[-h\left(\sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right)\right]\right\rangle\Big|_\gamma \tag{47}$$

with $\langle\ \rangle|_\gamma$ standing for an expectation value with the probability $P(\cdot|\gamma)$, and $\chi^c(\Delta\gamma)$ is the characteristic function of the event, "The set of the points of $\Delta\gamma$ and of the occupied points in $\Theta(\gamma)$ connected to $\Delta\gamma$ is a connected set." So we get

$$f_\mu(h) = \sum_{\gamma_1} [\exp(-h|\Delta\gamma|)]\left\langle \chi^c(\Delta\gamma)\chi(\Delta\gamma, \overline{\partial\gamma}) \exp\left(-h \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right)\right\rangle \tag{48}$$

where $\chi(\Delta\gamma, \overline{\partial\gamma})$ is the characteristic function of the event, "$\Delta\gamma$ is occupied and $\partial\gamma$ empty."

Now, we will prove that

$$\left\langle \chi^c(\Delta\gamma)\chi(\Delta\gamma, \overline{\partial\gamma}) \exp\left(-h \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right)\right\rangle$$
$$\geqslant \left\langle \chi^c(\Delta\gamma) \exp\left(-h \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right)\right\rangle \exp[-a(|\Delta\gamma| + |\partial\gamma|)] \tag{49}$$

where a is some constant. Then the divergence will follow as previously by choosing appropriate contours and from Jensen's inequality, since the probability for some point x in $\Theta(\gamma)$ to be connected to infinity is certainly less than the probability for x to be connected to $\Delta\gamma$, and so $\langle \chi_x^{\Delta\gamma} \rangle \geqslant P_\infty$.

In order to prove (49), for a given γ, we choose a box Λ, including γ, and then we will go to the thermodynamic limit. We forget for simplicity the possible boundary conditions; they can be handled similarly.

The function $\chi^c(\Delta\gamma) \exp(-h \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma})$ depends only on the configurations inside $\Theta(\gamma)$ and will be denoted for simplicity by g.

Now we have, by definition of the measure, that

$$\langle g\chi(\Delta\gamma, \overline{\partial\gamma})\rangle_\Lambda = \frac{1}{Z_\Lambda} \sum_{X\subset\Lambda\setminus(\Delta\gamma\cup\partial\gamma)} g(X) \exp[-\mu(|X| + \Delta\gamma) - U(X\cup\Delta\gamma)]$$

where $U(Y)$ is the energy of the set of points Y, and so

$$U(Y) = \sum_{\substack{Y'\subset Y\\ |Y'|\geqslant 2}} \Phi(Y')$$

Now let us introduce $W(X|Y)$ as

$$W(X|Y) = \sum_{\substack{T\subset X\cup Y\\ T\not\subset X}} \Phi(T)$$

where the sum runs over all subsets of the whole set of points, which contains at least one point of Y. Note that

$$|W(X|Y)| \leqslant \sum_{y\in Y} \sum_{\substack{T\subset(X\cup Y)\\ T\ni y}} |\Phi(T)| \leqslant |Y| D \tag{50}$$

So we can rewrite $\langle g\chi(\Delta\gamma, \overline{\partial\gamma}\rangle_\Lambda$ as

$$\langle g\chi(\Delta\gamma, \overline{\partial\gamma})\rangle_\Lambda = \frac{1}{Z_\Lambda} \sum_{X\subset\Lambda\setminus(\Delta\gamma\cup\partial\gamma)} g(X) e^{-\mu|X| - U(X)} e^{-\mu|\Delta\gamma| - W(X|\Delta\gamma)}$$

On the other hand, we can write $\langle g\rangle_\Lambda$ as

$$\langle g\rangle_\Lambda = \frac{1}{Z\Lambda} \sum_{X\subset\Lambda\setminus(\Delta\gamma\cup\partial\gamma)} e^{-\mu|X| - U(X)} g(X) \sum_{Y\subset(\Delta\gamma\cup\partial\gamma)} e^{-\mu|Y| - W(X|Y)}$$

It follows from these expressions, together, in particular, with (50), that

$$\langle g\chi(\Delta\gamma, \overline{\partial\gamma})\rangle_\Lambda \geqslant e^{-\mu|\Delta\gamma| - D(2|\Delta\gamma| + |\partial\gamma|)} 2^{-(|\Delta\gamma| + |\partial\gamma|)} \langle g\rangle_\Lambda \tag{51}$$

which gives (49) and so concludes the proof of the singularity of $f_\mu(h)$ at $h = 0$.

In the case of an equilibrium measure satisfying the FKG inequality, that is, for ferromagnetic systems,[19] we can say something more precise on the moments of the cluster size distribution; as a matter of fact, we can rewrite them as

$$\langle |C|^l\rangle = \sum_{\gamma_1} P(\gamma) \sum_{\substack{E\subset\Theta(\gamma)\\(E,\Delta\gamma)^c}} (\Delta\gamma + E)^l P(E, \overline{\partial E \cap \Theta(\gamma)}|\gamma) \tag{52}$$

$$= \sum_{\gamma_1} P(\gamma)\left\langle \chi^c(\Delta\gamma)\chi(\Delta\gamma, \overline{\partial\gamma})\left(|\Delta\gamma| + \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right)^l\right\rangle\bigg|_\gamma$$

$$= \sum_{\gamma_1} \left\langle \chi^c(\Delta\gamma)\chi(\Delta\gamma, \overline{\partial\gamma})\left(|\Delta\gamma| + \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right)^l\right\rangle \tag{53}$$

But, through (51), we have

$$\langle |C|^l\rangle \geqslant \sum_{\gamma_1} e^{-\alpha(|\Delta\gamma| + |\partial\gamma|)}\left\langle \chi^c(\Delta\gamma)\left(|\Delta\gamma| + \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right)^l\right\rangle \tag{54}$$

Now the functions $\chi^c(\Delta\gamma)$ and $\chi_x^{\Theta(\gamma)}$ are certainly increasing over the configurations; therefore the FKG inequality (which generalizes the Harris inequality to the case of ferromagnetic systems) tells us that

$$\langle |C|^l\rangle \geqslant \sum_{\gamma_1} e^{-\alpha(|\Delta\gamma| + |\partial\gamma|)}\langle\chi^c(\Delta\gamma)\rangle\left(\left\langle |\Delta\gamma| + \sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma}\right\rangle\right)^l \tag{55}$$

From (55), together with $\langle\chi_x^{\Delta\gamma}\rangle \geqslant P_\infty$, it follows now as in the noninteracting case that

$$\langle |C|^l\rangle \geqslant K^l\left(\frac{\nu}{\nu - 1}\, l\right)! \tag{56}$$

We can then state:

Theorem 4. For any lattice gas over $\mathbb{Z}^\nu$, $\nu > 1$, such that

$$\sum_{\substack{X\ni\{0\}\\X\subset\mathbb{Z}^\nu}} |\Phi(X)| < \infty$$

the following results hold in the percolative region:

(i) The cluster generating function $f_\mu(h)$ is singular at $h = 0$, and the P_n do not decay exponentially.
(ii) If, moreover, the system satisfies the FKG inequality, then

$$\langle |C|^n\rangle \geqslant K_1{}^n\left(\frac{\nu}{\nu - 1}\, n\right)!$$

These results generalize to realistic lattices different from $\mathbb{Z}^\nu$ as indicated.

As in the noninteracting case, our results on the moments $\langle |C|^n\rangle$ imply that if P_n behaves as $\exp(-\alpha n^\xi)$, then $P_n > \exp(-\beta n^{(\nu-1)/\nu})$. Arguments and numerical computations were given by Binder for such a behavior in the case of the clusters of the Ising model at low temperature.[20]

Remarks and Discussion

1. Our results have some interesting consequences for the usual ferromagnetic Ising model, with nearest neighbor interactions, in zero external field. Consider the positively magnetized equilibrium state obtained by taking positive boundary conditions. One knows[4] that in two dimensions, the plus spins percolate, whereas the minus do not in this state, when $T < T_c$. In three dimensions, the situation is expected to be the same at low temperatures. Our results imply that $P_n(+)$, the cluster distribution function for the clusters of plus spins, do not decay exponentially. On the other hand,[4] the probability of having a finite cluster of plus spins is less than the probability of the same cluster of minus spins; so $P_n(+) \leqslant P_n(-)$ and the $P_n(-)$ also do not decay exponentially.

2. Furthermore, if we impose a positive magnetic field H, the situation will be the same: The plus spins will percolate and the minus will not percolate for $T < T_c$ in two dimensions, and at low temperature in three dimensions. This follows from the monotonicity of P_∞ as a function of H.[21] Hence, in such a state, the probability for the spin at the origin to be minus is

$$P_H(\sigma_0 = -1) = \sum_{C_f \ni \{0\}} P_H(C_f)$$

where $P_H(C_f)$ is the probability of having a finite cluster C_f of minus spins.

If we approximate $P_H(C_f)$ by $[\exp(-H|C_f|)]P_{H=0}(C_f)$, which is an approximation similar to that of Andreev,[5] we get

$$P_H(\sigma_0 = -1) = \sum_{C_f \ni \{0\}} [\exp(-H|C_f|)]P_{H=0}(C_f)$$

which is the generating function for clusters of minus spins in the positively magnetized phase. Theorem 4 and the preceding remark prove that, in this approximation, the magnetization is singular at $H = 0$, at sufficiently low temperature. However, it remains to see if going beyond this approximation does not make the singularity disappear.

3. Surprisingly, in the interacting case, at low temperature, one can easily find a reasonable approximation to $P_{H=0}(C_f)$ which reproduces the singular behavior, but not in the noninteracting case. Indeed, $P_{H=0}(C_f)$ can be approximated reasonably well by $\exp[-2Jl(C_f)]$, where $l(C_f)$ is the number of bonds between the points of the cluster and the outside, since this is the first term in a low-temperature expansion of $P_{H=0}(C_f)$. And if we restrict the summation to the most compact clusters, which are cubes, then we easily get

the singularity. This was essentially the kind of approximation made by Fisher,[5] who could conjecture from this basis the existence of a singularity for the free energy at the boundary of a phase transition region.

In the noninteracting case, however (i.e., for the usual percolation model), or at high temperature, if we approximate the cluster generating function by restricting the summation to the most compact clusters, we do not recover our result. Indeed, the most compact clusters, the cubes, appear in the non-interacting case with probability $p^{l^\nu} q^{\partial l}$, where $\partial l = (l + 2)^\nu - l^\nu - 2\nu l^{\nu-2}$. Therefore

$$f^{(a)}(h) = \sum_{l=2\nu}^{\infty} q^{\partial l}(pe^{-h})^{l^\nu}$$

has a singularity at $h = \log p$, but is analytic for $h \geqslant \log p$ and in particular at $h = 0$. In fact, a reasonable approximation should take into account an average over clusters with given external boundary, as suggested by the central limit theorem in the concluding remark of this paper.

5. FURTHER RESULTS FOR LARGE CONCENTRATIONS

We want now to prove here some other results (see Theorem 5) which can be obtained for the noninteracting percolation problem when the concentration is large and which yield, together with Section 4, a detailed knowledge of the behavior of the various functions involved. Here again we will restrict our attention to the site problem on $\mathbb{Z}^\nu$, but adaptations to bond problem and to other lattices are readily obtained according to the methods indicated in Section 4.

First, we will show that for large concentrations, the moments are finite and $f_p(h)$ is infinitely differentiable. Let us consider Eq. (29). On the right-hand side, we can certainly bound $\chi^c(\Delta\gamma)$ and $\chi_x^{\Delta\gamma}$ by 1, which yields the following estimate:

$$\langle |C|^l \rangle \leqslant \sum_{\gamma_1} p^{|\Delta\gamma|} q^{|\partial\gamma|} [V(\gamma)]^l \tag{57}$$

Now if $|\gamma|$ denotes the length of a contour, that is, the number of bonds joining one point in $V(\gamma)$ to one point of $\mathbb{Z}^\nu \backslash V(\gamma)$, we see that

$$|\Delta\gamma| \geqslant |\gamma|/(z - 1), \qquad |\partial\gamma| \geqslant |\gamma|/(z - 1)$$

On the other hand, a Peierls estimate tells us that the number of contours of length k is smaller than B^k, where B is some constant. So (57) leads to the bound

$$\langle |C|^l \rangle \leqslant \sum_{k} B^k (pq)^{k/(z-1)} (k^{\nu/(\nu-1)})^l \tag{58}$$

The right-hand side of (58) is convergent for p larger than some $p(\nu)$ and implies through tedious but quite direct computations that

$$\langle |C|^l \rangle \leqslant K^l \left(\frac{\nu}{\nu - 1} l \right)! \tag{59}$$

The bound (59) implies that for $p > p(\nu)$, the function $f_p(h)$ is infinitely differentiable at $h = 0$, and, together with the results of Section 4, that

$$K_1{}^l \left(\frac{\nu}{\nu - 1} l \right)! \leqslant \langle |C|^l \rangle \leqslant K_2{}^l \left(\frac{\nu}{\nu - 1} l \right)! \tag{60}$$

The singularity at $h = 0$ is then an essential one, at least for large concentrations. As a matter of fact, one should expect $f_p(h)$ to be infinitely differentiable for any p larger than p_c. In this respect we note that such a result is implied by Russo's work[22] in the two-dimensional case.

Let us study now the P_n. We want to prove that for large concentrations they behave as $\exp(-\alpha n^{(\nu-1)/\nu})$. Let us first get the upper bound. Using arguments of the type used in Section 4, we see that the P_n can be expressed as

$$P_n/n = \sum_{\substack{\gamma_1 \\ \Delta\gamma \leqslant n;\, V(\gamma) \geqslant n}} p^{|\Delta\gamma|} q^{|\partial\gamma|} \left\langle \chi^c(\Delta\gamma) \chi\left\{ \sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma} = n - |\Delta\gamma| \right\} \right\rangle_{\Theta(\gamma)} \tag{61}$$

where the sum runs over all shapes of contours such that $\Delta\gamma$ is less than or equal to n and the number of points inside γ is greater than or equal to n, and where $\chi\{\sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma} = n - |\Delta\gamma|\}$ is the characteristic function of the event, "there are exactly $n - |\Delta\gamma|$ occupied points in $\Theta(\gamma)$ connected to $\Delta\gamma$."

Hence, by majorizing again the χ by 1, we get

$$P_n/n \leqslant \sum_{\substack{\gamma_1 \\ V(\gamma) \geqslant n}} p^{|\Delta\gamma|} q^{|\partial\gamma|} \tag{62}$$

Using again a Peierls estimate, together with the remark that for a contour of length k, we have $V(\gamma) \leqslant k^{\nu/(\nu-1)}$, we obtain

$$P_n/n \leqslant \sum_{k > n^{(\nu-1)/\nu}} K^k p^{k/z} \tag{63}$$

which in turn implies that for $p > p(\nu)$, the P_n satisfy

$$P_n/n \leqslant \exp[-\alpha(p) n^{(\nu-1)/\nu}] \tag{64}$$

We would like now to prove a lower bound on the P_n. In fact, this will be more complicated than the upper one. Let us first introduce the mean number Q_n of clusters of size larger than or equal to n:

$$Q_n = \sum_{m \geqslant n} P_m/m \tag{65}$$

Inequality (64) shows that for large concentrations,

$$Q_n \leqslant \exp[-\alpha(p)n^{(\nu-1)/\nu}] \tag{66}$$

and we will, as a first step, prove a lower bound of the same type for the Q_n.

Lemma. For $p > p(\nu)$, the Q_n satisfy a lower bound:

$$Q_n \geqslant \exp[-b(p)n^{(\nu-1)/\nu}]$$

for some constant $b(p)$.

The Q_n can be expressed from (61) as

$$Q_n = \sum_{\substack{\gamma_1 \\ V(\gamma)\geqslant n}} p^{|\Delta\gamma|}q^{|\partial\gamma|}\left\langle \chi^c(\Delta\gamma)\chi\left\{\sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma} \geqslant n - |\Delta\gamma|\right\}\right\rangle_{\Theta(\gamma)} \tag{67}$$

where the characteristic function inside the brackets runs on the event, "there are at least $n - |\Delta\gamma|$ occupied points in $\Theta(\gamma)$ connected to $\Delta\gamma$."

We restrict now the summation to contours γ' such that $\Delta\gamma$ is a connected set; hence $\chi^c(\Delta\gamma) \equiv 1$, and, moreover, we have

$$\left\langle \chi\left\{\sum_{x\in\Theta(\gamma)} \chi_x^{\Delta\gamma} \geqslant n - |\Delta\gamma|\right\}\right\rangle_{\Theta(\gamma)} \geqslant \left\langle \chi\left\{\sum_{x\in\Theta(\gamma)} \chi_x^{\infty} \geqslant n - |\Delta\gamma|\right\}\right\rangle \tag{68}$$

since the probability of finding at least $n - |\Delta\gamma|$ points of $\Theta(\gamma)$ connected to infinity is certainly less than the probability of finding at least $n - |\Delta\gamma|$ points of $\Theta(\gamma)$ connected to $\Delta\gamma$. Here χ_x^{∞} denotes the characteristic function of the event, "x is connected to infinity." Furthermore, we can then drop the subscript $\Theta(\gamma)$ for the average value.

At this state, we introduce the variables S_γ:

$$S_\gamma = \sum_{x\in\Theta(\gamma)} (\chi_x^{\infty} - P_\infty) \tag{69}$$

and we restrict the summation in (67) to the cubes of volume l^ν

$$\begin{aligned} Q_n \geqslant \sum_{l\geqslant n^{1/\nu}} &\{\exp[-\alpha'(p)l^{\nu-1}]\} \\ &\times \operatorname{Prob}\{S_\gamma \geqslant n - [l^\nu - (l-2)^\nu] - P_\infty(l-2)^\nu\} \end{aligned} \tag{70}$$

Now let us choose l such that

$$(n/P_\infty)^{1/\nu} \leqslant l - 2 \leqslant (n/P_\infty)^{1/\nu} + 1 \tag{71}$$

and so $l > n^{1/\nu}$ since $P_\infty \leqslant 1$. Furthermore, in view of (71), we have

$$n - P_\infty(l-2)^\nu - [l^\nu - (l-2)^\nu] \leqslant -[l^\nu - (l-2)^\nu] \leqslant -2\nu(l-2)^{\nu-1} \tag{72}$$

On the other hand, $\mathrm{Prob}\{S_\gamma \geqslant m\}$ is a decreasing function of m, by definition, and then

$$\begin{aligned}&\mathrm{Prob}\{S_\gamma \geqslant n - [l^\nu - (l-2)^\nu] - P_\infty(l-2)^\nu\}\\ &\quad\geqslant \mathrm{Prob}\{S_\gamma \geqslant -2\nu(l-2)^{\nu-1}\} = 1 - \mathrm{Prob}\{S_\gamma < -2\nu(l-2)^{\nu-1}\}\end{aligned} \tag{73}$$

Since the variable S_γ is centered, we can apply the Bienaymé–Tschebycheff inequality, which states that

$$\mathrm{Prob}(|X| > a) \leqslant \langle (X/a)^2 \rangle \tag{74}$$

Hence

$$\begin{aligned}&\mathrm{Prob}\{S_\gamma < -2\nu(l-2)^{\nu-1}\}\\ &\quad\leqslant \mathrm{Prob}\{|S_\gamma| > 2\nu(l-2)^{\nu-1}\} \leqslant [2\nu(l-2)^{\nu-1}]^{-2}\langle S_\gamma^2\rangle\end{aligned} \tag{75}$$

Note that, moreover,

$$\langle S_\gamma^2 \rangle = \sum_{\substack{x\in\Theta_l\\ y\in\Theta_l}} [\langle \chi_x^\infty \chi_y^\infty\rangle - \langle \chi_x^\infty\rangle\langle\chi_y^\infty\rangle] \tag{76}$$

and we have, by the Harris[14] or FKG[19] inequality, that

$$\langle \chi_x^\infty \chi_y^\infty\rangle - \langle \chi_x^\infty\rangle\langle\chi_y^\infty\rangle \geqslant 0$$

Then from (76) we get

$$\langle S_\gamma^2\rangle \leqslant (l-2)^\nu \chi \tag{77}$$

with χ defined by

$$\chi = \sum_{y\in\mathbb{Z}^\nu} \langle \chi_0^\infty \chi_y^\infty\rangle - \langle \chi_0^\infty\rangle\langle\chi_y^\infty\rangle \tag{78}$$

The inequalities (70), (73), and (75) imply that

$$Q_n \geqslant \{\exp[-\alpha n^{(\nu-1)/\nu}]\}\left[1 - \frac{\chi}{(2\nu)^2}\frac{1}{(l-2)^{\nu-2}}\right] \tag{79}$$

This in turn ensures the desired lower bound on Q_n, if the quantity inside the second pair of square brackets is positive. This is the case when $\nu \geqslant 3$ if n is large enough and when $\nu \geqslant 2$ for any n if χ is small enough. As a matter of fact, we will show now that χ is finite and χ is arbitrarily small when p is large. This will end the proof of the lemma and will allow us to derive a lower bound for the P_n.

So we consider the quantity χ. As a matter of fact, we will prove that

$$\chi_{0,y} \equiv \langle \chi_0^\infty \chi_y^\infty\rangle - \langle \chi_0^\infty\rangle\langle\chi_y^\infty\rangle \leqslant K(q)[qK(\nu)]^{d(0,y)} \tag{80}$$

with $K(q)$ small when $q = 1 - p$ is small, which will yield the desired result, together with the previous remark that $\chi_{0,y} \geqslant 0$.

If χ_x^f denotes the characteristic function of the event, "x belongs to a finite cluster," then

$$\chi_x^\infty = 1 - \chi_x^f \tag{81}$$

So using (81) in the definition of $\chi_{0,y}$ leads to

$$\chi_{0,y} = \langle \chi_0^f \chi_y^f \rangle - \langle \chi_0^f \rangle \langle \chi_y^f \rangle \tag{82}$$

In terms of clusters, (82) can be rewritten as

$$\chi_{0,y} = \sum_{C \in 0 \text{ and } y} P(C) + \sum_{\substack{C_1 \ni 0;\, C_2 \ni y \\ C_1 \cap C_2 = \varnothing}} P(C_1 \text{ and } C_2) - \sum_{C_1 \ni 0;\, C_2 \ni y} P(C_1)P(C_2) \tag{83}$$

Finally, we may use that if $\partial C_1 \cap \partial C_2 = \varnothing$, then $P(C_1 \text{ and } C_2) = P(C_1)P(C_2)$, and so

$$\chi_{0,y} = \sum_{C \ni 0 \text{ and } y} P(C) + \sum_{\substack{C_1 \ni 0;\, C_2 \ni y \\ C_1 \cap C_2 = \varnothing;\, \partial C_1 \cap \partial C_2 \neq \varnothing}} [P(C_1 \text{ and } C_2) - P(C_1)P(C_2)]$$
$$- \sum_{\substack{C_1 \ni 0;\, C_2 \ni y \\ C_1 \cap C_2 \neq \varnothing}} P(C_1)P(C_2) \tag{84}$$

Each of these four terms will satisfy a bound of the form (80). As an example, in the first one, which is the probability that 0 and y belong to the same finite cluster, there exists necessarily a closed contour $\mathscr{L}$, encircling both 0 and y, and such that $\partial\mathscr{L}$ is empty. This implies, through arguments already used for the upper bound on the P_n, the desired decay property. The other terms are bounded in a similar way. For the next two, either one of the C_1 and C_2 encircles the other, and the proof is the same, or C_1 and C_2 are external to each other with at least a common point in their boundary and one considers the two closed contours passing through this point and encircling one the origin, the other the point y. The last term, for which the clusters themselves intersect, is bounded also in the same way.

This concludes the proof of the bound (80), which in turn implies the lemma on the lower bound for the Q_n.

We are now going to use this result to obtain a lower bound for the P_n. At this step, we know that the Q_n satisfy, for p larger than some $p(\nu)$,

$$\exp[-\beta(p)n^{(\nu-1)/\nu}] \leqslant Q_n = \sum_{m \geqslant n} P_m/m \leqslant \exp[-\beta'(p)n^{(\nu-1)/\nu}] \tag{85}$$

If A is some given integer, let us consider

$$Q_n - Q_{A^\nu n} = \sum_{n \leqslant m < A^\nu n} P_m/m \tag{86}$$

From (85), it follows that

$$Q_n - Q_{A^\nu n} \geqslant \exp[-\beta(p)n^{(\nu-1)/\nu}] - \exp[-\beta'(p)A^{\nu-1}n^{(\nu-1)/\nu}] \tag{87}$$

We fix now A, for p given, such that $\beta'(p)A^{\nu-1} > 2\beta(p)$, and then (87) yields the existence of $\beta''(p)$ such that

$$Q_n - Q_{A^\nu n} \geqslant \exp[-\beta''(p)n^{(\nu-1)/\nu}] \tag{88}$$

Now from (88) and (86) we see that for any n there exists some $k(n)$ such that

$$n \leqslant k(n) < A^\nu n \tag{89}$$

$$\frac{P_{k(n)}}{k(n)} \geqslant \frac{1}{A^\nu - 1}\frac{1}{n}\exp[-\beta''(p)n^{(\nu-1)/\nu}] \tag{90}$$

If we consider now the intervals

$$I_1 = [1, A^\nu[, I_2 = [A^\nu, A^{2\nu}[,..., I_l = [A^{l\nu}, A^{(l+1)\nu}[,...$$

we will denote, for each l, by $k(l)$ one of the integers of the interval I_l for which (90) holds.

We have then obtained a sequence $k(l)$ which has the desired lower bound. We will prove now that the properties of this sequence, together with the upper-additivity of $\log(P_n/n)$ (see the lemma in Section 4) implies a good lower bound on the P_n.

We will use that for any n, there exist integers $a(n, l)$, $l_0(n)$, and $n_0(n)$ satisfying the four following properties:

$$n = \sum_{l \leqslant l_0} a(n, l)k(l) + n_0(n) \tag{91}$$

$$A^{(l_0+1)\nu} \leqslant n < A^{(l_0+2)\nu} \tag{92}$$

$$0 \leqslant a(n, l) < A^{2\nu} \tag{93}$$

$$0 \leqslant n_0(n) < A^\nu \tag{94}$$

The decomposition (91)–(94) generalizes the decomposition of an integer onto a nondecimal basis. This is necessary because the $k(l)$ are not necessarily of the form r^l, but only belong to the sequence of intervals I_l. This decomposition can be proved by induction as follows. A given n belongs to some interval I_{l_0+1}, defining hence $l_0(n)$, satisfying (92). Then one defines $a(n, l_0)$ such that

$$a(n, l_0)k(l_0) \leqslant n < [a(n, l_0) + 1]k(l_0) \tag{95}$$

From (95) it follows that

$$a(n, l_0) \leqslant n/k(l_0) \leqslant A^{(l_0+2)\nu}/A^{l_0\nu} = A^{2\nu}$$

and $a(n, l_0)$ then satisfies (93); moreover, (95) implies that $n - a(n, l_0) \times k(l_0) < k(l_0)$, and so $n - a(n, l_0) \times k(l_0)$ can be decomposed in turn by induction into the $k(l)$ for $l < l_0$, defining successively $a(n, l_0 - 1)$,

$a(n, l_0 - 2), \ldots$, all satisfying (93). At the end the remainder is called $n_0(n)$ and satisfies (94).

Now in view of (91) and of the property $P_{n+m}/(n + m) \geqslant (P_n/n)(P_m/m)$ (see the lemma in Section 4), we get

$$\log \frac{P_n}{n} \geqslant \sum_{l \leqslant l_0} \log \frac{P_{a(n,l)\,k(l)}}{a(n, l)k(l)} + \log \frac{P_{n_0(n)}}{n_0(n)} \tag{96}$$

which in turn yields

$$\log \frac{P_n}{n} \geqslant \sum_{l \leqslant l_0} a(n, l) \log \frac{P_{k(l)}}{k(l)} + \log \frac{P_{n_0(n)}}{n_0(n)} \tag{97}$$

and finally, if α denotes $(\nu - 1)/\nu$, we get

$$\frac{1}{n^\alpha} \log \frac{P_n}{n} \geqslant \sum_{l \leqslant l_0} a(n, l) \frac{k(l)^\alpha}{n^\alpha} \frac{1}{k(l)^\alpha} \log \frac{P_{k(l)}}{k(l)} + \frac{n_0(n)^\alpha}{n^\alpha} \frac{1}{n_0(n)^\alpha} \log \frac{P_{n_0(n)}}{n_0(n)} \tag{98}$$

But, on the one hand, $n_0(n)$ is always bounded by A^ν from property (94), and then

$$\frac{1}{n_0(n)^\alpha} \log \frac{P_{n_0(n)}}{n_0(n)} > -K$$

for some constant K. On the other hand, the sequence $k(l)$ satisfies by construction the property (90), and so, for any n and l,

$$\frac{1}{k(l)^\alpha} \log \frac{P_{k(l)}}{k(l)} > -K'$$

for some constant K'.

Hence, in view of (98), we get

$$\frac{1}{n^\alpha} \log \frac{P_n}{n} \geqslant -K - K' \sum_{l \leqslant l_0(n)} a(n, l) \frac{k(l)^\alpha}{n^\alpha} \tag{99}$$

In view of (93), (92), and of the fact that $k(l)$ belongs to the interval I_l, this provides

$$\frac{1}{n^\alpha} \log \frac{P_n}{n} \geqslant -K - K' \frac{A^{2\nu}}{A^{(l_0+1)(\nu-1)}} \sum_{l \leqslant l_0(n)} A^{(l+1)(\nu-1)} \tag{100}$$

and we see that the right-hand side of (100) is bounded below for any l_0 by some constant $-K''$ depending only on A, that is, on p. Hence we have proved that for $p > p(\nu)$ and for any n

$$\frac{1}{n^{(\nu-1)/\nu}} \log \frac{P_n}{n} \geqslant -K'' \tag{101}$$

which yields the desired result for the lower bound on the P_n.

Before regrouping these results in a theorem, we wish to show how the bound (64) implies a regularity property of $f_p(h)$ at $h = 0$, namely the Borel-summability, and that $f_p(h)$ possesses then an integral representation. This result has been obtained in collaboration with G. Parisi.

Let us first introduce the following function $G(t)$, which will be proved to be the Borel transform of $f_p(h)$:

$$G(t) = \frac{1}{\sqrt{\pi}} \sum_n \cos[(tn)^{1/2}] \frac{P_n}{n} \tag{102}$$

By virtue of $|\cos[(tn)^{1/2}]| < \exp(\sqrt{n}\,|\mathrm{Im}\,\sqrt{t}|)$, and from (64), the series (102) will be absolutely convergent in the whole complex plane and then entire in the case $\nu \geqslant 3$, and absolutely convergent and analytic in the case $\nu = 2$ in the following parabola, which includes the whole positive real axis:

$$(\mathrm{Im}\, t)^2 < 4\alpha(p)^2(\mathrm{Re}\, t) + 4\alpha(p)^4$$

Now we will prove that

$$f_p(h) = \int_0^\infty G(th)e^{-t} \frac{dt}{\sqrt{t}} \tag{103}$$

In fact, if we insert the expression (102) into (103), the double summation is absolutely convergent for $t \geqslant 0$ and we can permute them. Hence

$$\int_0^\infty G(th)e^{-t} \frac{dt}{\sqrt{t}} = \frac{1}{\sqrt{\pi}} \sum_n \frac{P_n}{n} \int_0^\infty \cos[(tnh)^{1/2}]\, e^{-t} \frac{dt}{\sqrt{t}} \tag{104}$$

But the integral on the right-hand side of (104) can be computed explicitly, and the result is $\sqrt{\pi}\, e^{-hn}$, which concludes the proof.

We can now regroup all these results:

Theorem 5. For the bond and site percolation problems over $\mathbb{Z}^\nu$ or over the other realistic lattices, $\nu > 1$, besides the results of Theorem 3, the following properties hold for p larger than some $p_0(\nu)$:

(i) $f_p(h)$ is infinitely differentiable at $h = 0$, and, moreover, is Borel-summable. The integral representation (103) holds, with $G(t)$ satisfying the described properties.

(ii) The moments of the cluster size distribution behave as

$$K_1^l\left(\frac{\nu}{\nu - 1}\, l\right)! \leqslant \langle |C|^l \rangle \leqslant K_2^l\left(\frac{\nu}{\nu - 1}\, l\right)!$$

(iii) The cluster distribution function satisfies for all n

$$\exp[-\alpha(p)n^{(\nu-1)/\nu}] \leqslant P_n/n \leqslant \exp[-\alpha'(p)n^{(\nu-1)/\nu}]$$

Finally, we think that the techniques developed in Sections 4 and 5 should allow one to prove results analogous to Theorem 5 in the case of the interacting percolation problem. We will not develop here our work in that direction. In the next section we explain some of the phenomena proved in this paper, within the framework of a central limit theorem.

6. CONCLUSION: PERCOLATION AND THE CENTRAL LIMIT THEOREM

We discuss the previous results from a qualitative point of view. We have seen in this paper that all the results concerning $f(h)$, $\langle |C|^n \rangle$, and P_n crucially depend on the behavior of the average volume of the clusters for a given external boundary.

So let us consider for a given external boundary γ the variable $S_\gamma = \sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma}$, which gives the number of occupied points inside $\Theta(\gamma)$ connected to $\Delta\gamma$. Now, $\langle S_\gamma \rangle$ is precisely the average number of points of the interior of the clusters with external boundary γ.

Now in various situations, for example, for $p \neq p_c$, in the noninteracting case, one expects that a central limit theorem should hold. This means that the distribution of the variable

$$[\Theta(\gamma)]^{-1/2} \sum_{x \in \Theta(\gamma)} (\chi_x^{\Delta\gamma} - \langle \chi_x^{\Delta\gamma} \rangle) \tag{105}$$

should tend to that of a Gaussian centered at the origin when γ tends to infinity.

Since $\langle \chi_x^{\Delta\gamma} \rangle \to P_\infty$, this implies that

$$\operatorname{Prob}\left(\sum_{x \in \Theta(\gamma)} \chi_x^{\Delta\gamma} = m \right) \sim \frac{1}{[2\pi\Theta(\gamma)\chi]^{1/2}} \exp\left(-\frac{[m - \Theta(\gamma)P_\infty]^2}{2\Theta(\gamma)\chi} \right) \tag{106}$$

which is a "δ-function" in the limit of large γ.

Hence, if we consider as an example the cluster distribution function P_n, we have in view of (61)

$$P_n/n \sim \sum_{\gamma_1} p^{|\Delta\gamma|} q^{|\partial\gamma|} \, \delta_{n, \Delta\gamma + \Theta(\gamma)P_\infty} \tag{107}$$

and so the external contours γ contributing to the P_n satisfy

$$\Delta\gamma \sim \partial\gamma \sim n \qquad \text{if} \quad P_\infty = 0 \tag{108}$$

$$\Delta\gamma \sim \partial\gamma \sim (n/P_\infty)^{(\nu-1)/\nu} \qquad \text{if} \quad P_\infty \neq 0 \tag{109}$$

Equations (108) and (109) explain, in view of (107), the qualitatively different behavior of the P_n outside and inside the percolative region.

In fact, we expect that the methods of this paper, in particular some of those in Section 5, should allow one to get a central limit theorem at least for low and high concentrations. It would certainly be of interest to have a general discussion of this point.

Finally, these remarks show that a good estimate for the percolation quantities should take into account not only special clusters (compact,...), but an average value over the clusters with same external boundary, such as the one described by (107)–(109).

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REFERENCES

1. S. R. Broadbent and J. M. Hammersley, *Proc. Camb. Phil. Soc.* **53**:629 (1957).
2. C. M. Fortuin and P. W. Kasteleyn, *Physica* **57**:535 (1972); C. M. Fortuin, *Physica* **58**:393 (1972), **59**:545 (1972).
3. M. Miyamoto, *Comm. Math. Phys.* **44**:169 (1975).
4. A. Coniglio, C. R. Nappi, F. Peruggi, and L. Russo, *Comm. Math. Phys.* **51**:315 (1976); *J. Phys. A* **10**:205 (1977).
5. A. F. Andreev, *Sov. Phys.—JETP* **18**:1415 (1964); M. E. Fisher, *Physics* **3**:255 (1967).
6. H. Kunz and B. Souillard, *Phys. Rev. Lett.* **40**:133 (1978).
7. F. Y. Wu, *J. Stat. Phys.* **18**:115 (1978).
8. H. Kunz and F. Y. Wu, *J. Phys. C* **11**:L1 (1978).
9. E. H. Lieb, unpublished.
10. L. K. Runnels and J. L. Lebowitz, *J. Stat. Phys.* **14**:525 (1976).
11. M. Schwartz, to appear in *Phys. Rev. B* (1978).
12. D. Ruelle, *Statistical Mechanics* (Benjamin, New York, 1969).
13. M. E. Fisher and J. W. Essam, *J. Math. Phys.* **2**:609 (1961).
14. T. E. Harris, *Proc. Camb. Phil. Soc.* **56**:13 (1960).
15. D. Stauffer, *Z. Physik B* **25**:391 (1976).
16. A. Flamang, *Z. Physik B* **28**:47 (1977).
17. D. Stauffer, *J. Phys. C* **8**:172 (1975); G. R. Reich and P. L. Leath, *J. Phys. C* **11**:1155 (1978).
18. D. Stauffer, *J. Stat. Phys.* **18**:125 (1978).
19. C. M. Fortuin, J. Ginibre, and P. W. Kasteleyn, *Comm. Math. Phys.* **22**:89 (1971).
20. K. Binder, *Ann. Phys. (NY)* **98**:390 (1976); *J. Stat. Phys.* **15**:267 (1976).
21. J. L. Lebowitz and O. Penrose, *J. Stat. Phys.* **16**:321 (1977).
22. L. Russo, A note on percolation, Preprint.